1 (5 points). Is the graph of the function $x \mapsto |x|$ the trace of a smooth map $F: \mathbb{R} \to \mathbb{R}^2$? If yes, give an example of such a map; if not, explain why not.

2 (15 points). Fix a real number $c \neq 0$. Show that the set of $n \times n$ matrices with determinant equal to $c$ is a manifold. (You may use the fact that the derivative of the determinant function at the identity is the trace function.)

3 (15 points). Prove or disprove: every vector field on $S^2$ must have at least two distinct zeros.

4 Let $\alpha$ be a $k$-form on $M$ and let $X$ and $Y$ be vector fields on $M$.
   (i) (10 points). Prove that
   \[ L_X(i_Y \alpha) = i_Y(L_X \alpha) + i_{(L_X Y)}\alpha. \]
   (ii) (5 points). Prove that $L_{[X,Y]}\alpha = 0$ whenever $L_X \alpha = 0$ and $L_Y \alpha = 0$.

5 (15 points). Prove that every map from $S^2$ to $T^2$ has zero degree.

6 (10 points). Construct a closed 3-manifold whose fundamental group is the three-element cyclic group $\mathbb{Z}/3\mathbb{Z}$.

7 (10 points). Let $g$ be a metric on $S^2$ with curvature less than or equal to one. Show that the area of $S^2$ is greater than or equal to $4\pi$.

8 (15 points). Let $M$ be a closed hypersurface in $\mathbb{R}^3$. Show that $M$ cannot have negative curvature at every point. (Hint: consider the point on $M$ farthest from the origin.)