Geometry–Topology Prelim, Spring 2009

1 (5 points). Is the graph of the function $x \mapsto |x|$ the trace of a smooth map $F\colon \mathbb{R} \to \mathbb{R}^2$? If yes, give an example of such a map; if not, explain why not.

2 (15 points). Fix a real number $c \neq 0$. Show that the set of $n \times n$ matrices with determinant equal to c is a manifold. (You may use the fact that the derivative of the determinant function at the identity is the trace function.)

3 (15 points). Prove or disprove: every vector field on S^2 must have at least two distinct zeros.

4 Let α be a k-form on M and let X and Y be vector fields on M.

(i) (10 points). Prove that

$$L_X(i_Y\alpha) = i_Y(L_X\alpha) + i_{(L_XY)}\alpha.$$

(ii) (5 points). Prove that $L_{[X,Y]}\alpha = 0$ whenever $L_X\alpha = 0$ and $L_Y\alpha = 0$.

5 (15 points). Prove that every map from S^2 to T^2 has zero degree.

6 (10 points). Construct a closed 3-manifold whose fundamental group is the three-element cyclic group $\mathbb{Z}/3\mathbb{Z}$.

7 (10 points). Let g be a metric on S^2 with curvature less than or equal to one. Show that the area of S^2 is greater than or equal to 4π.

8 (15 points). Let M be a closed hypersurface in $\mathbb{R}^3$. Show that M cannot have negative curvature at every point. (Hint: consider the point on M farthest from the origin.)