TOPOLOGY/GEOMETRY PRELIMINARY EXAM
UCSC Mathematics Graduate Program
Spring 2010

Instruction: *Write your answer in precise and concise language. Your writing directly reflects the clarity of your thinking. Therefore writing is part of your scores. Show all of your work in an organized way. Write your name code in all of your answer sheets.*

[**1**] (15 points) Let $Y = \bigvee^3 S^1$ be a one point union of three circles. Let a, b, c be the based oriented loops in Y tracing each of these three circles. Let X be a cell complex obtained by attaching a 2-cell e^2 by an attaching map $\varphi : S^1 \to Y$ given by $\varphi = abca^{-1}bc^{-1}$.

(1) Compute the integral homology of X.

(2) Using the classification theory of surfaces, show that X is a nonorientable surface of genus 3, that is, X is homeomorphic to a connected sum $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

[**2**] (15 points) Let a smooth map $f : \mathbb{R}P^2 \to \mathbb{R}^4$ be given by

$$f([x; y; z]) = (x^2 - y^2, yz, xz, xy),$$

where $\mathbb{R}P^2$ is regarded as the quotient of the unit sphere S^2 by the antipodal map. Is f a smooth embedding?

[**3**] (15 points) Let X, Y be smooth vector fields on a manifold M. Prove that $[X, Y] = 0$ if and only if $\varphi(t)_* Y = Y$, where $\varphi(t)$ is the local flow of X.

[**4**] (10 points) Let $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an invertible linear map, and let $f : \mathbb{R}P^n \to \mathbb{R}P^n$ be the induced map on the real projective space.

(1) Suppose $n = 2m, m \geq 1$. Show that f has at least one fixed point on $\mathbb{R}P^{2m}$.

(2) Suppose $n = 2m - 1, m \geq 1$. Find an example of a linear map F whose induced map f does not have a fixed point on $\mathbb{R}P^{2m-1}$.

[**5**] (15 points) Let Σ_g be an orientable compact surface of genus g without boundary.

(1) A cyclic group of order $g - 1$ freely acts on Σ_g with the quotient space Σ_2. Describe this $(g-1)$-to-1 covering map $\Sigma_g \to \Sigma_2$.

(2) Prove that there exists a covering map $\Sigma_{g+1} \to \Sigma_{h+1}$ if and only if h divides g.

[**6**] (10 points) Let M be a compact connected simply-connected smooth 4-dimensional manifold without boundary. Show that every smooth vector field on M has a zero.

[**7**] (10 points) Calculate the Gaussian curvature of the surface $z = 3x^2 - 5y^2$ at the point $(0, 0, 0)$.

[**8**] (10 points) A Riemannian manifold is called *homogeneous* if the isometry group acts transitively. Prove that a homogeneous manifold is geodesically complete.