Geometry and Topology Preliminary Exam

Spring 2011

1. **2.** **3.** **4.** **5.** **6.** **7.** **8.** **9.**

Instructions: Do *seven* of the problems. Indicate which seven problems you want graded by circling the appropriate numbers on the line above. All problems are worth the same number of points.

[**1**] The join $X * Y$ of two topological spaces X and Y is

$$X * Y = X \times I \times Y / \sim,$$

where the equivalence relation $\sim$ is given by $(x_1, 1, y) \sim (x_2, 1, y)$, and $(x, 0, y_1) \sim (x, 0, y_2)$ for any $x, x_1, x_2 \in X, y, y_1, y_2 \in Y$. (Basically, what we are doing is the following: for any point $x \in X$ and $y \in Y$, join them by the unit interval I. You do this for every $x \in X$ and $y \in Y$, and take the union of all of these intervals, hence the name "join" of X and Y.) It is convenient to denote the image of $\{x\} \times \{t\} \times \{y\}$ in the join $X * Y$ formally by $(1-t)x + t\,y$. We identify $y \in Y$ with $0\,x + y$ corresponding to $t = 1$ in the join. This embeds Y into the join. Similarly, X is embedded in the join at the other end where $t = 0$.

(a) Show that the complement $X * Y - Y$ of Y in $X * Y$ deformation retracts to its subspace X.

(b) Show that $S^n * S^m = S^{n+m+1}$ for any $n, m \geq 0$.

(c) Let $S^k \subset S^n$ be the standard inclusion, $k \leq n-1$. Show that $S^n - S^k$ is homotopy equivalent to S^{n-k-1}.

[**2**] Let M be a smooth, compact, oriented manifold with boundary. Prove that there is no smooth retraction of M onto its boundary ∂M, i.e., no smooth map $f \colon M \to \partial M$ such that such that $f|_{\partial M} = id$.

[**3**] Let M be the set of all mutually orthogonal ordered pairs of unit vectors in $\mathbb{R}^3$.

(a) Show that M is a smooth manifold.

(b) Is M orientable?

[**4**] Calculate the Gaussian curvature of the surface $z = \cos^2 x + \sin^2 y$ at the point $(0, 0, 1)$.

[5] Let v_0, v_1, v_2, v_3 be ordered points in general position in $\mathbb{R}^3$. Let $[v_0, v_1, v_2, v_3]$ be a tetrahedron obtained by taking the convex hull of these four points. Let X be a topological space obtained by identifying all four faces by affine linear maps preserving orders of vertices.

(a) Give a cell decomposition of X.

(b) Compute the homology group of X.

[6] Let L, M, and N be manifolds and let $p : L \to M$ and $q : L \to N$ be smooth maps. Show that if p is a surjective submersion and $f : M \to N$ satisfies $f \circ p = q$, then f is smooth.

[7] Consider the form

$$\omega = |x|^{-n} \sum_{i=1}^{n} x_i \, dx_1 \wedge \ldots \wedge \hat{dx_i} \wedge \ldots \wedge dx_n$$

on $\mathbb{R}^n \setminus \{0\}$. Show that ω is closed but not exact.

[8] Compute the homology group of the space X obtained from a torus $S^1 \times S^1$ by attaching a Möbius band M via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.

[9] (a) Consider the vector field $V(x) = Ax + b$ on $\mathbb{R}^n$, where A is an $n \times n$ matrix and $b \in \mathbb{R}^n$. Find the flow of V.

(b) Given $V_j(x) = A_j x + b_j$, $j = 1, 2$, as in (a), compute $[V_1, V_2]$.