Geometry and Topology Prelim. Spring, 2012

1. M is a smooth oriented compact surface and c is a smooth embedded circle in M.

Prove or disprove. There is a Riemannian metric on M such that c is a geodesic.

2. (a) Prove that a connected manifold is path-connected.

(b) Let M be a connected manifold. Show for any two points x and y in M there exists a diffeomorphism φ sending x to y which is isotopic to the identity. (A diffeomorphism φ is called isotopic to the identity if there exists a family of diffeomorphisms φ_t, smooth in $t \in [0, 1]$, such that φ_0 is the identity and $\varphi_1 = \varphi$.)

3. Prove or disprove: There is no continuous map $F : S^2 \to S^1$ which is the identity on the equator, $S^1 \subset S^2$.

4. Prove or disprove. There is a submersion $S^5 \to \mathbb{C}P^2$.

5. Consider the function $f(y) = \cos y$ on $\mathbb{R}$. Show that there is a diffeomorphism φ from a neighborhood V of zero in $\mathbb{R}$, equipped with coordinate x, onto another neighborhood of zero such that $(\varphi^* f)(x) = 1 - x^2$.

6. Show that there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}_2) \to H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ if $n > m \geq 1$.

7. Consider the upper half plane with its standard hyperbolic metric $\frac{1}{y^2}(dx^2 + dy^2)$. For k a fixed real number, compute the Laplacian of the function y^k relative to this metric.