Preliminary Examination in Geometry and Topology

Spring 2013

All problems have the same point value.

1. For each positive integer n, define $X_n \subseteq \mathbb{R}^2$ by $X_n = \{(n,x) \mid x \in \mathbb{R}\}$, i.e. X_n is the vertical line through $(n,0)$. Define $f : \cup_n X_n \to \mathbb{R}^2$ by

$$f(n,x) = \begin{cases} (0,x) & n = 1 \\ \left(\frac{1}{n}, x\right) & n > 1 \end{cases}$$

   Is f an immersion? Is f an embedding?

2. Consider the distribution Δ on $\mathbb{R}^3$ given by

$$\Delta_{(x,y,z)} = \operatorname{span}\left\{ y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right\}.$$

   (a) Show that this distribution is integrable.

   (b) Find a 1-form β such that $\Delta_{\mathbf{x}}$ is the nullspace of $\beta(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^3$.

   (c) Describe the maximal integral submanifolds of Δ.

3. Let S be a compact orientable surface of genus two.

   (a) Compute the singular homology of S.

   (b) Compute the deRham cohomology of S.

   You may use without proof any results you know about the (co)homology of spheres and tori, as long as they are stated precisely.

4. Let $G = GL(2,\mathbb{R})$. Given matrices A and $B \in \mathbb{R}^{2\times 2}$, let X and Y be the left-invariant vector fields on G satisfying $X(I_2) = A$ and $Y(I_2) = B$, where I_2 denotes the 2×2 identity matrix, and let β be the left-invariant 1-form satisfying

$$\beta(I_2)(C) = \operatorname{trace} C \qquad \text{for any} \quad C \in \mathbb{R}^{2\times 2}.$$

   Compute the function $\iota_Y \iota_X \mathrm{d}\beta$.

5. Let X be the quotient space of an annulus obtained by identifying antipodal points on the outer circle, and identifying points on the inner circle which are 120 degrees apart. Find $\pi_1(X)$.

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6. Let S be an ellipsoid with principal axes of length a, b, and c satisfying $c > b > a > 0$.

   (a) What are the maximum and minimum of the Gaussian curvature of S?

   (b) For every point p at which the Gaussian curvature achieves its maximum or minimum, describe two distinct closed geodesics passing through p.

   Hint: Invoke an appropriate theorem; don't directly solve the geodesic equations.

7. Let $m(t)$ be an integral curve of a complete smooth vector field X on a smooth manifold M. Let $g : M \to \mathbb{R}$ be smooth, $I \subseteq \mathbb{R}$ be an open interval, and $\tau : I \to \mathbb{R}$ be a smooth function satisfying $\tau'(t) = g(m(\tau(t)))$ for all $t \in I$.

   (a) Show that $\widetilde{m} := m \circ \tau$ is an integral curve of the vector field $g\,X$.

   (b) Show that $g\,X$ is complete or provide an example for which $g\,X$ is not complete.