GEOMETRY AND TOPOLOGY PRELIMINARY EXAM

UCSC Mathematics, Spring 2014

[1] Consider the following vector fields on $\mathbb{R}^2$.

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \qquad Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad Z = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}.$$

(1) Draw the picture of directed flow lines generated by these vector fields.

(2) Show that the three dimensional vector space $V = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z$ is closed under bracket operation. Write down all commutation relations among them.

[2] Define a map $F : S^2 \to \mathbb{R}^4$ by

$$F(x, y, z) = (x^2 - y^2, xy, yz, zx).$$

Show that F descends to a smooth embedding from $\mathbb{R}P^2$ into $\mathbb{R}^4$.

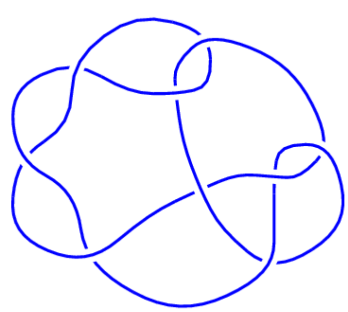
[3] Prove that the complex projective space $\mathbb{CP}^n$ is orientable for all n.

[4] (1) Prove that every smooth map T^3 to S^4 is smoothly homotopic to a constant map, i.e., to a map sending T^3 into one point.

(2) Show that every smooth map S^4 to T^3 is smoothly homotopic to a constant map.

[5] Consider a unit sphere and a torus so that the torus touches the sphere along the equator of the sphere filling the hole of the torus. Call this space X. Let x_0 be a point on the equator of the sphere. Compute the funbdamental group $\pi_1(X, x_0)$ and the homology group $H_*(X; \mathbb{Z})$.

[6] Let X be the compact surface spanning a knot given below. There are many possible such surfaces. Choose one and draw its picture. Let $\widehat{X}$ be the closed surface obtained by capping the boundary of X. Identify the surface $\widehat{X}$. Namely, find the genus and the orientability of $\widehat{X}$.



[7] Let (M, g) be Riemannian manifold and let f be a smooth function. Calculate the gradient $\nabla f = g^{-1}(df)$ in local coordinates.

[8] Show that a reparametrization $t \to \alpha(f(t))$ of a nonconstant geodesic α is again a geodesic if and only if f has the form $f(t) = a\,t + b$ for some constants a and b.