Preliminary Examination. Geometry and Topology. UC Santa Cruz. Spring 2015.

1. X is a smooth vector field defined on a smooth n-dimensional manifold M and $p \in M$ is a point of M. Suppose that $X(p) \neq 0$. *Prove or find a counterexample:* there is a smooth vector field Y on M such that $[X, Y] = X$ holds true in some neighborhood of p.

2. Let M and N be oriented, compact, connected smooth manifolds, and $F, G : M \to N$ be homotopic diffeomorphisms between them. Show that F and G are either both orientation-preserving or both orientation-reversing. (You may assume that the homotopy is itself a homotopy through diffeomorphisms.)

3. Let $M = M^3$ be an oriented 3 dimensional Riemannian manifold, endowed with its standard Riemannian volume form.

A) Define the divergence $div(X)$ of a vector field X on M.

B) Prove or find a counterexample. If $div(X) \neq 0$ then there exists a smooth positive function f on M such that $div(fX) = 0$ everywhere.

4.Let $\Gamma \subset SU(2)$ be a finite non-Abelian simple group. Compute $H^2(SU(2)/\Gamma, \mathbb{R})$.

5.Let c be a smooth embedded closed curve in the plane $\mathbb{R}^2$.

A) Define the curvature κ of c. (It is function along c.)

B) Prove or find a counterexample: If c lies strictly inside the open unit disc then there is a point p along c where the curvature $\kappa(p)$ satisfies $|\kappa(p)| > 1$.

6. Let $f : S^n \to S^n$ be a degree 0 map. Show that there exist points $x, y \in S^n$ such that $f(x) = x$ and $f(y) = -y$.

7. Evaluate the integral $\int_{S^2} z\, dx \wedge dy$.