
1. $X$ is a smooth vector field defined on a smooth $n$-dimensional manifold $M$ and $p \in M$ is a point of $M$. Suppose that $X(p) \neq 0$. Prove or find a counterexample: there is a smooth vector field $Y$ on $M$ such that $[X,Y] = X$ holds true in some neighborhood of $p$.

2. Let $M$ and $N$ be oriented, compact, connected smooth manifolds, and $F,G : M \to N$ be homotopic diffeomorphisms between them. Show that $F$ and $G$ are either both orientation-preserving or both orientation-reversing. (You may assume that the homotopy is itself a homotopy through diffeomorphisms.)

3. Let $M = M^3$ be an oriented 3 dimensional Riemannian manifold, endowed with its standard Riemannian volume form.
   A) Define the divergence $\text{div}(X)$ of a vector field $X$ on $M$.
   B) Prove or find a counterexample. If $\text{div}(X) \neq 0$ then there exists a smooth positive function $f$ on $M$ such that $\text{div}(fX) = 0$ everywhere.

4. Let $\Gamma \subset SU(2)$ be a finite non-Abelian simple group. Compute $H^2(SU(2)/\Gamma, \mathbb{R})$.

5. Let $c$ be a smooth embedded closed curve in the plane $\mathbb{R}^2$.
   A) Define the curvature $\kappa$ of $c$. (It is function along $c$.)
   B) Prove or find a counterexample: If $c$ lies strictly inside the open unit disc then there is a point $p$ along $c$ where the curvature $\kappa(p)$ satisfies $|\kappa(p)| > 1$.

6. Let $f : S^n \to S^n$ be a degree 0 map. Show that there exist points $x,y \in S^n$ such that $f(x) = x$ and $f(y) = -y$.

7. Evaluate the integral $\int_{S^2} z \, dx \wedge dy$. 

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