**GEOMETRY AND TOPOLOGY PRELIMINARY EXAM**

**Spring 2016**

**[1]** Let $p \in M$ be a regular point of a smooth function $f\colon M \to \mathbb{R}$. Prove that there is a coordinate system $(x_1, \dots, x_n)$ near $p$ such that $f(x) = f(p) + x_1$.

**[2]** Let $f\colon \mathbb{C}P^n \to \mathbb{R}$ be given by

$$f(z_0, \dots, z_n) = \frac{\lambda_0|z_0|^2 + \dots + \lambda_n|z_n|^2}{|z_0|^2 + \dots + |z_n|^2},$$

where all coefficients $\lambda_j$ are distinct. Show that $f$ is a smooth function and find its critical points. (There are exactly $n+1$ critical points.)

**[3]** Prove or disprove: The Lie group $SO(3)$ admits a closed one-form is not exact.

**[4]** A Riemannian metric on a surface has the form $du^2 + f(u)^2dv^2$ where $f(u) = \cosh(ku)$, $k > 0$ a parameter, $u, v$ coordinates on the surface. Compute the Gaussian curvature for this metric.

**[5]** Given two oriented triangles $[v_0, v_1, v_2]$ and $[w_0, w_1, w_2]$, identify edges as follows:

$$[v_0, v_1] \sim [w_0, w_2], \quad [v_0, v_2] \sim [w_0, w_1], \quad [v_1, v_2] \sim [w_1, w_2].$$

(1) Identify the resulting surface (that is, find orientability and genus).

(2) Describe its chain complex and compute its homology using the above $\Delta$-complex structure.

**[6]** Let $X$ be a path connected topological space with a base point $x_0 \in X$. Let $\pi_1(X, x_0)$ be its fundamental group, and let $[S^1, X]$ be its free homotopy classes of loops. There is a canonical map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ sending based homotopy classes to free homotopy classes. Show that for $[f], [g] \in \pi_1(X, x_0)$, $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

**[7]** Let $(\Omega, \phi)$ be a local coordinate chart in a Riemannian manifold $M^n$, where

$$\phi = (x^1, x^2, \cdots, x^n) : \Omega \to R^n.$$

Let $\nabla$ be the Levi-Civita covariant differentiation and let

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

a) Given a vector field $v = \sum_{i=1}^n a^i(x) \frac{\partial}{\partial x^i}$, calculate its covariant derivative $\nabla_{\frac{\partial}{\partial x^k}} v$.

b) Let $\tilde{\gamma}(t) = (x^1(t), x^2(t), \cdots x^n(t))$ be the coordinate of a curve $\gamma$ in $M$. Show that its velocity vector is

$$\gamma'(t) = \sum_{i=1}^n \frac{dx^i}{dt} \frac{\partial}{\partial x^i}.$$

c) A curve is said to be a geodesic if $\nabla_{\gamma'(t)}\gamma'(t) = 0$. Show that the equation of a geodesic in local coordinate is

$$\frac{d^2x^k}{dt^2} + \sum_{i,j=1}^n \Gamma_{ij}^k \frac{dx^i}{dt} \frac{dx^j}{dt} = 0$$

for $k = 1, 2, \cdots, n$.