GEOMETRY AND TOPOLOGY PRELIMINARY EXAM

Spring 2018

[1] Let x, y denote standard coordinates on $\mathbb{R}^2$.

(1) Find smooth vector fields U, V on $\mathbb{R}^2$ such that $[U, V] = \frac{\partial}{\partial x}$ and $U(0,0) = 0$

(2) Show that there is no such pair of vector fields U, V such that $[U, V] = \frac{\partial}{\partial x}$, $U(0,0) = 0$, and $V(0,0) = 0$.

[2] On an n-dimensional manifold M^n there is given a smooth non-vanishing one-form ω such that $\omega \wedge d\omega = 0$. Show that in a neighborhood on any point p on M there exists smooth coordinates $x_1, x_2, \ldots, x_n$ and a smooth positive function f such that $\omega = f dx_1$ in this neighborhood. [Hint; the Frobenius integrability theorem will be useful]

[3] Let $C \subset \mathbb{R}^2$ be a smooth closed embedded curve. We can then parameterize c with a map $c : \mathbb{R} \to \mathbb{R}^2$ satisfying $c(s + L) = c(s)$ where L is the length of the curve. Let $N(s)$ denote the outward pointed normal to C. Define the map $C \times \mathbb{R} \to \mathbb{R}^2$ by sending (P, t) to $P + tN(s)$, where $P = c(s) \in C$ and $t \in \mathbb{R}$. Compute the critical points of this map and establish a relation between these critical points and the curvature of the curve C.

[4] (1) Let S be a closed smooth oriented surface of genus 2 in $\mathbb{R}^3$, and let $i : S \to \mathbb{R}^3$ be the inclusion map. Sketch the construction of an explicit differential 2-form $\omega \in \Omega^2(S)$ such that

$$\int_S \omega = 2018.$$

Does there exist a 2-form $\eta \in \Omega^2(\mathbb{R}^3)$ such that $d\eta = 0$ and $\iota^*(\eta) = \omega$?

(2) Let $F : \mathcal{M}^n \to \mathcal{N}^n$ be a map between closed connected oriented n-manifolds. Suppose that there exists a smooth n-form ω on $\mathcal{N}$ such that $\int_{\mathcal{M}} F^*\omega \neq 0$. Prove that F is surjective.

[5] For each $k \in \mathbb{R}$, define $\omega_k \in \Omega^2(\mathbb{R}^3 \setminus \{0\})$ by

$$\omega_k = (x^2 + y^2 + z^2)^k (x dy \wedge dz + y dz \wedge dx + z dx \wedge dy)$$

(1) Compute $d\omega_k$. Find the unique k for which ω_k is closed.

(2) Compute $\int_{S^2} \omega_k$, where S^2 denotes the unit sphere. (Note that this is independent of k.)

(3) Let $\Sigma \subset \mathbb{R}^3$ be the compact surface obtained by revolving the ellipse given by $x = 0, y^2 + 2z^2 = 7$ about the line $x = 0, y = 15$. For the unique k found above, compute $\int_\Sigma \omega_k$.

[6] The fundamental group $\pi_1(X, x_0)$ is the set of homotopy classes of base point preserving maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps without conditions on base points. Then there exists a map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring base points.

1. Show that Φ is onto if X is path connected.
2. Show that when X is path connected, Φ induces a bijection between the set of conjugacy classes of $\pi_1(X, x_0)$ and the set of free homotopy classes $[S^1, X]$.

[7] Let Σ_2 be the genus 2 closed orientable surface and let C be a circle around the "neck" of Σ_2. Does there exist a continuous map $r : \Sigma_2 \to C$ such that $r|_C = 1_C$? If yes, construct such a map. If not, prove the nonexistence.