Geometry–Topology Prelim, Winter 2010

1 (10 points). Is there a C^∞-map $f\colon [0,\,1] \to \mathbb{R}^2$ whose image is the square $[0,\,1] \times [0,\,1]$?

2 (15 points). Let $f\colon \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4$ be the map

$$f([x,y,z]) = (x^2 - y^2,\, yz,\, xz,\, xy),$$

where $\mathbb{R}\mathrm{P}^2$ is viewed as the quotient of the unit sphere in $\mathbb{R}^3$ by the antipodal map. Is f an embedding?

3 (15 points). Let X and Y be vector fields on a smooth manifold Prove that $[X,Y]=0$ if and only if $\varphi^t_* Y = Y$, where φ^t is the local flow of X.

4 (10 points). Let $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ be differential forms and let X be a vector field on M. Prove that

$$i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^k \alpha \wedge i_X\beta.$$

5 (10 points). Let Σ_g denote the sphere with g handles. (Thus, e.g., $\Sigma_1 = \mathbb{T}^2$.) Describe a $(g-1)$-to-1 covering map $\Sigma_g \to \Sigma_2$, where $g \geq 2$.

6 (15 points). Prove that on a four-dimensional compact, connected, simply connected, closed manifold, every vector field has a zero.

7 (10 points). Find the curvature of the surface $z = 3x^2 - 5y^2$ at the point $(0,0,0)$.

8 (15 points). A Riemannian manifold is called *homogeneous* if the isometry group acts transitively. Prove that a homogeneous manifold is geodesically complete.