Prelim, Winter, 2012, in Differential Geometry and Topology. All problems are worth the same point value.

1. Consider vector fields $v_k = x^k \frac{\partial}{\partial x}$ on $\mathbb{R}$, where $k \geq 0$.
(a) Find the Lie bracket $[v_i, v_j]$.
(b) Do the flows of v_i and v_j commute?
(c) Find the flow of v_2.
(d) Is v_2 complete on $\mathbb{R}$?

2. (a) Prove that antipodal map on $S^n \to S^n$ given by $x \mapsto -x$ on the unit sphere S^n in $\mathbb{R}^{n+1}$ is orientation preserving if and only if n is odd.
(b) Prove that the real projective space $\mathbb{R}P^n$ is orientable if and only n is odd.

3. A helicoid in $\mathbb{R}^3$ is parameterized $(s,t) \mapsto (s\cos t, s\sin t, t)$. Please compute the helicoid's:
a) first fundamental form
b) second fundamental form
c) Gaussian curvature
d) mean curvature
as functions of s, t.

4. A three-manifold M is framed by vector fields X, Y, Z. Let $\theta^1, \theta^2, \theta^3$ be the dual coframe to this frame. Suppose we know the Lie bracket relations

$$[X,Y] = -2Z,$$
$$[Y,Z] = X,$$
$$[Z,X] = Y.$$

Let $c^i_{jk} = -c^i_{kj}$ be the structure functions for this co-frame; thus: $d\theta^i = \Sigma c^i_{jk}\theta^j \wedge \theta^k$. (For example: $d\theta^1 = c^1_{23}\theta^2 \wedge \theta^3 + c^1_{31}\theta^3 \wedge \theta^1 + c^1_{12}\theta^1 \wedge \theta^2$.)
a) Compute the structure functions.
b) True or False: Must M be a Lie group?

5. On the manifold $\mathbb{R}^2 \setminus \{0\}$, please:
a) Find a closed one-form that is not exact.
b) Show that any compactly supported closed one-form is exact. [The support of a differential form is the point set on which it does not vanish.]

6. Prove that every smooth map from S^3 to T^3 has degree 0.