Preliminary Examination in Geometry and Topology

Winter 2013

All problems have the same point value.

1. Prove that for almost all pairs \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n\), with \(x \neq y\) and \(n > 1\), the line through \(x\) and \(y\) does not pass through the origin.

2. Show that Green’s Theorem for regions in \(\mathbb{R}^2\) with boundary consisting of a disjoint union of one or more circles is a consequence of Stokes’ Theorem.

3. Let \(f : M_1 \to M_2\) be a continuous map between oriented compact manifolds without boundary. The degree of \(f\) is the integer \(\deg f\) defined by \(f_*([M_1]) = (\deg f)[M_2]\), where \([M_1], [M_2]\) are the integral fundamental homology classes. Show that every map \(S^{k+\ell} \to S^k \times S^n\) necessarily has degree zero.

4. Define the vector fields \(X\) and \(Y\) on \(\mathbb{R}^2\) by

\[
X(x, y) := y^2 \frac{\partial}{\partial x} \quad \text{and} \quad Y(x, y) := x^2 \frac{\partial}{\partial y}.
\]

Show that \(X\) and \(Y\) are complete, but \(X + Y\) is not.

5. Prove or provide a counterexample: Suppose \(M\) is a smooth manifold and and \(X\) and \(Y\) are smooth vector fields on \(M\). If \(X\) and \([X, Y]\) are tangent to a smooth submanifold \(N\) of \(M\) and the restrictions of \(X\) and \([X, Y]\) to \(N\) are nonzero, then \(Y\) is also tangent to \(N\).

6. Let \(T\) be the torus constructed by rotating the circle of radius 1 in the \(x-z\) plane centered at \((0, 0, 2)\) about the \(x\) axis. Which points in \(T\) have zero Gaussian curvature? What are the maximum and minimum of the Gaussian curvature on \(T\)?

7. Show that the Euler characteristic of a compact connected nontrivial Lie group is zero.