Preliminary Examination in Geometry and Topology

Winter 2013

All problems have the same point value.

1. Prove that for almost all pairs $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, with $x \neq y$ and $n > 1$, the line through x and y does not pass through the origin.

2. Show that Green's Theorem for regions in $\mathbb{R}^2$ with boundary consisting of a disjoint union of one or more circles is a consequence of Stokes' Theorem.

3. Let $f : M_1 \to M_2$ be a continuous map between oriented compact manifolds without boundary. The degree of f is the integer $\deg f$ defined by $f_*([M_1]) = (\deg f)[M_2]$, where $[M_1], [M_2]$ are the integral fundamental homology classes. Show that every map $S^{k+\ell} \to S^k \times S^\ell$ necessarily has degree zero.

4. Define the vector fields X and Y on $\mathbb{R}^2$ by

$$X(x, y) := y^2 \frac{\partial}{\partial x} \qquad \text{and} \qquad Y(x, y) := x^2 \frac{\partial}{\partial y}.$$

   Show that X and Y are complete, but $X + Y$ is not.

5. Prove or provide a counterexample: Suppose M is a smooth manifold and and X and Y are smooth vector fields on M. If X and $[X, Y]$ are tangent to a smooth submanifold N of M and the restrictions of X and $[X, Y]$ to N are nonzero, then Y is also tangent to N.

6. Let T be the torus constructed by rotating the circle of radius 1 in the x–z plane centered at $(0, 0, 2)$ about the x axis. Which points in T have zero Gaussian curvature? What are the maximum and minimum of the Gaussian curvature on T?

7. Show that the Euler characteristic of a compact connected nontrivial Lie group is zero.