## GEOMETRY AND TOPOLOGY PRELIMINARY EXAM

**Winter 2014**

**[1]** Define a map $F : S^2 \to \mathbb{R}^4$ by

$$F(x, y, z) = (x^2 - y^2, xy, yz, zx).$$

Show that $F$ descends to a smooth embedding from $\mathbb{R}P^2$ into $\mathbb{R}^4$.

**[2]** Prove that the complex projective space $\mathbb{CP}^n$ is orientable for all $n$.

**[3]** Consider the following vector fields on $\mathbb{R}^2$.

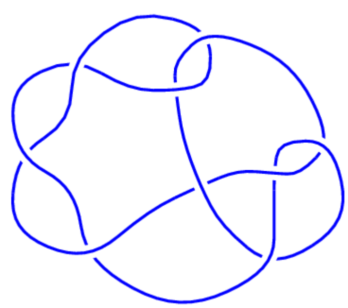
$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \qquad Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad Z = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}.$$

(1) Draw the picture of flows generated by these vector fields.

(2) Show that the three dimensional vector space $V = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z$ is closed under bracket. Write down all commutation relations.

**[4]** Consider a unit sphere and a torus so that the torus touches the sphere along the equator of the sphere. Call this space $X$. Let $x_0$ be a point on the equator. Compute the funbdamental group $\pi_1(X, x_0)$ and the homology group $H_*(X; \mathbb{Z})$.

**[5]** Let $X$ be the compact surface spanning a knot given below. There are many possible such surfaces. Choose one and draw its picture. Let $\widehat{X}$ be the closed surface obtained by capping $X$. Identify the surface $\widehat{X}$. Namely, find the genus and the orientability of $\widehat{X}$.



**[6]** Let $(M, g)$ be Riemannian manifold and let $f$ be a smooth function. Calculate the gradient $\nabla f = g^{-1}(df)$ in local coordinates.

**[7]** Show that a reparametrization $t \to \alpha(f(t))$ of a nonconstant geodesic $\alpha$ is again a geodesic if and only if $f$ has the form $f(t) = a\,t + b$ for some constants $a$ and $b$.