Geometry and Topology Prelim, UCSC, Winter 2015

1. Let μ be the two-form given by $\mu = \frac{1}{y^2} dx \wedge dy$ on the upper half-plane $y > 0$.

(a) Express the vanishing of the Lie derivative $L_v \mu$ of μ with respect to the vector field $v = v_1(x, y)\frac{\partial}{\partial x} + v_2(x, y)\frac{\partial}{\partial y}$ as a first order linear PDE for the pair of components v_1 and v_2.

(b) Verify that $v = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ satisfies this PDE.

(c) Find the flow Φ_t of the vector field v from (b). Verify that $\Phi_t^* \mu = \mu$ by a direct computation, using your expression for Φ_t. Explain how this equality relates to (a) and (b).

2. Let M be a compact, connected, and orientable smooth 6-dimensional manifold without boundary. Let α and β be two smooth 2-forms on M. Show that there is a point of M at which $d\alpha \wedge d\beta = 0$.

3. Let $M = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0\}$ and define a distribution D on M by

$$D_{(x,y,z)} = \operatorname{span}\left\{ y\,\frac{\partial}{\partial x} + x\,y\,\frac{\partial}{\partial z},\, x\,\frac{\partial}{\partial y} + x\,y\,\frac{\partial}{\partial z} \right\}.$$

Show that D is integrable and describe the maximal integral submanifolds of D.

4. Consider the space X formed by deleting two disjoint disks from the plane.

(a) What is the rank of the 1st homology group of X (with $\mathbb{Z}$-coefficients) ?

(b) Describe the fundamental group of X.

(c) Draw a closed curve in X which is homologically trivial but homotopically non-trivial.

(d) Repeat (a) and (b) for the space obtained by removing three disjoint disks from $\mathbb{R}^2$.

5. Does the real projective plane $\mathbb{R}P^2$ admit an embedding into $\mathbb{R}^4$? Describe such an embedding or show that it does not exist.

6. Using the cup product structure in the cohomology ring, show that there is no continuous map $\mathbb{C}P^n \to \mathbb{C}P^m$ inducing a nontrivial homomorphism $H^2(\mathbb{C}P^m; \mathbb{Z}) \to H^2(\mathbb{C}P^n; \mathbb{Z})$ if $n > m$.

Over please!

7.

(a) Let M be a Riemannian manifold and let $\sigma\colon M \to M$ be an isometry. Show that if the fixed point set of σ is a connected one-dimensional submanifold of M, then it is the image of a geodesic.

(b) Let M be a smooth surface of revolution in $\mathbb{R}^3$. A meridian on M is a connected component of the intersection of M and a plane through the axis of revolution. Prove that if M is endowed with the Riemannian metric induced by the Euclidean inner product on $\mathbb{R}^3$, then a meridian is the image of a geodesic.