Geometry Topology Prelim: Winter 2017

[**1**] True or False? 'If X and Y are smooth vector fields on a manifold and p is a point of the manifold and X(p) = 0 and Y(p) =0, then [X,Y](p) = 0.' Give a proof or counterexample according to your answer.

[**2**] Let $F\colon X \to Y$ be an (onto) submersion such that Y is connected and for every point $y \in Y$ the inverse image $F^{-1}(y)$ is connected.
(1) Prove that X is connected.
(2) Is the converse true? If X is connected must all inverse images be connected?

[**3**] Consider the distribution

$$\mathrm{Ker}\,(x\,dx + y\,dy + v\,du + u\,dv) \cap \mathrm{Ker}\,(x\,dx + u\,du + v\,dv)$$

on the subset $y \neq 0$ of $\mathbb{R}^4$ with coordinates (x, y, u, v). Is this distribution involutive?

[**4**] Consider an annulus. Identify antipodal points along the outer circle, and also identify antipodal points along the inner circle. Calculate the fundamental group of this identification space X. Is this space X a compact topological manifold? If not, prove it. If yes, identify this space among the surfaces appearing in the classification theory.

[**5**] (1) Consider a unit square $I = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$. Identify the opposite edges by translation. Calculate the homology of the resulting surface.
(2) Consider a 3-dimensional unit cube $K = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x, y, z \leq 1\}$. Identify opposite faces by translation. Calculate the homology of the resulting 3-dimensional manifold.

[**6**] Consider the upper half plane with standard hyperbolic metric $g = \frac{dx^2+dy^2}{y^2}$. Give the coordinate formula for the gradient ∇f of a function $f(x, y)$ with respect to this metric.

[**7**] Suppose that (M, g) is a Riemannian manifold.
(1) Define the exponential map.
(2) Prove that the differential of the exponential map at the origin of the tangent space at a point is the identity.