Geometry and Topology Prelim, UCSC, Winter 2018

1. Let ω be a non-vanishing n-form on an n-dimensional manifold and η an $(n-1)$-form. Show that there exists a unique vector field X such that $i_v\omega = \eta$.

2. Let $F\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an invertible linear map, and let $f\colon \mathbb{RP}^n \to \mathbb{RP}^n$ be the induced map on the real projective space.

(a) Suppose n is even. Show that f has at least one fixed point on $\mathbb{RP}^n$, i.e., $f(p) = p$ for some $p \in \mathbb{RP}^n$.
(b) Suppose n is odd. Find an example of a linear map F such that f has no fixed points on $\mathbb{RP}^n$.

3. Suppose that X and Y are smooth everywhere linearly independent vector fields on a manifold M and that $[X, Y] = Y$. True or False: There exists, in a neighborhood of any point of M, a smooth positive function f such that $[X, fY] = 0$. If 'True' give a proof; if 'False' give a counterexample.

4. In this problem X is a complete vector field on a manifold M and φ_t, $t \in \mathbb{R}$, is the flow of X.

(a) True or False: If $lim_{t\to+\infty}\varphi_t(p) = p$ for all $p \in M$ then $X = 0$.
(b) True or False: It cannot happen that there exists a point p_0 such that for all $p \in M$ we have $lim_{t\to\infty}\varphi_t(p) = p_0$. If 'True' give a proof; if 'False' give a counterexample.

5. In a 3-simplex $[v_0, v_1, v_2, v_3]$, identify all four faces by affine linear maps preserving orders of vertices (thus preserving the orientation of faces). Let X be the resulting topological space. Compute the homology of X with integer coefficients and with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

6. Suppose S is a compact surface with two boundary components and $\chi(S) = -1$.

(a) Identify S.
(b) Compute the fundamental group of S.

7. Consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ equipped with the flat metric $g = dx^2 + dy^2$, where (x, y) are the coordinates on $\mathbb{R}^2$. Does $(\mathbb{T}^2, g)$ admit an isometric immersion in $\mathbb{R}^3$? Justify your answer: construct such an immersion or prove that it does not exist.