GEOMETRY AND TOPOLOGY PRELIMINARY EXAM

Winter 2019

[1] Let $[x; y; z; w]$ be homogeneous coordinates in $\mathbb{R}P^3$ and consider the subset X defined by the homogeneous polynomial equation

$$x^2 + y^2 - z^2 = 0$$

Prove that X is a smooth manifold except at one (singular) point. Can you describe the shape of X qualitatively?

[2] Identify $\mathbb{R}^2$ with coordinates (x, y) with $\mathbb{C}$, with coordinate $z = x + iy$. Likewise, identify a copy of $\mathbb{R}^2$ with coordinates u, v with $\mathbb{C}$ with coordinate $w = u + iv$. Let a function $f : \mathbb{R}^2 \setminus \{(1, 0), (-1, 0)\} \longrightarrow \mathbb{R}^2$ be given by

$$f(z) = \frac{1}{z-1} - \frac{1}{\overline{z}+1}.$$

(1) Show that f extends to a smooth map $\overline{f} : S^2 \to S^2$, where S^2 (or $\mathbb{C}P^1$) is the one-point compactification of $\mathbb{R}^2$ (or $\mathbb{C}$).
(2) Compute the degree of f.

[3] Let $\mathbb{R}^2$ be parametrized by (u, v), and let R^3 be parametrized by (x, y, z). In $\mathbb{R}^2$, let $U = (0, \pi) \times (0, 2\pi)$, and let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(u, v) = \big(\sin(u)\cos(v), \sin(u)\sin(v), \cos(u)\big)$$

(1) Compute $f^*(x), f^*(y) f^*(z), f^*(dx), f^*(dy)$ and $f^*(dz)$.
(2) Consider the differential form $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ on $\mathbb{R}^3$. Compute $\int_U f^*(\omega)$.

[4] Consider a vector field in $\mathbb{R}^2$ whose value at $x \in \mathbb{R}^2$ is

$$V(x) = v_1(x)\frac{\partial}{\partial x_1} + v_2(x)\frac{\partial}{\partial x_2}$$

for v_i smooth functions, and let f_1, f_2 be smooth functions $\mathbb{R} \to \mathbb{R}$.

(1) Calculate the Lie derivative $\mathcal{L}_V\left(f_1(x_1)dx_1 \otimes \frac{\partial}{\partial x_1} + f_2(x_2)dx_2 \otimes \frac{\partial}{\partial x_2}\right)$ and find the condition on f_1, f_2 arising from the constraint $\mathcal{L}_V = 0$.
(2) Calculate the Lie derivative $\mathcal{L}_V\left(f_1(x_1)dx_1 \otimes dx_1 + f_2(x_2)dx_2 \otimes dx_2\right)$ and find the condition on f_1, f_2 arising from the constraint $\mathcal{L}_V = 0$. How does this compare with the condition from the previous problem?

[5] Let $T = S^1 \times S^1$ be a torus with a base point (x_0, y_0). Let $X = \mathbb{R}P^2 \cup T$ be a space obtained by identifying $\mathbb{R}P^1 \subset \mathbb{R}P^2$ with $S^1 \times \{y_0\} \subset T$.

(1) Compute the fundamental group of X.
(2) Let T' be a punctured torus, that is, T' is T minus one point $p \notin \mathbb{R}P^1$. Let $X' = \mathbb{R}P^2 \cup T'$ be the space obtained by a similar construction. What is the fundamental group of X'?

[6] Let D^3 be the 3-dimensional closed unit ball. Let $f : D^3 \to D^3$ be a continuous map. Show that f has a fixed point.

[7] Let M be the boundary of a compact (not necessarily convex) polyhedron in $\mathbb{R}^3$ (with straight edges and flat faces). At each vertex v of M, define its vertex angle defect $d(v)$ by

$$d(v) = 2\pi - \sum(\text{face angles at } v),$$

where the summation runs over all faces incident to the vertex v, and by "face angle" we mean the angle at v within each incident face. Prove the following discrete version of Gauß-Bonnet Theorem.

$$\sum_v d(v) = 2\pi\chi(M).$$