GEOMETRY AND TOPOLOGY PRELIMINARY EXAM

Fall 2013

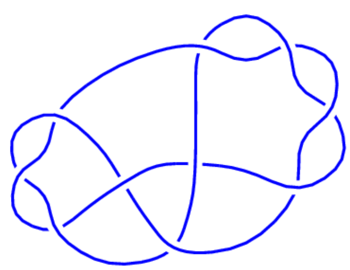
[**1**] Let M be a smooth manifold and let $F : M \times \mathbb{R} \to M$ be a smooth mapping satisfying $F(m, 0) = m$ and $F(m, t + s) = F(F(m, s), t)$ for all $m \in M$ and all $s, t \in \mathbb{R}$. Show that there is a unique smooth vector field X whose flow is F.

[**2**] Consider the following vector fields v and w on $\mathbb{R}^2$.

$$v = (x^2 + y^2)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \qquad w = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

(a) Are these vector fields complete?
(b) Find the Lie bracket $[v, w]$. Do the flows of these vector fields commute?

[**3**] Let X be the compact surface spanning a knot given below. There are many possible such surfaces. Choose one and draw its picture. Let $\widehat{X}$ be the closed surface obtained by capping X. Identify $\widehat{X}$. Namely, find the genus and the orientability of $\widehat{X}$.



[**4**] Is there an embedding of $S^1 \times S^2$ into $\mathbb{R}^4$?

[**5**] Let $Y = \bigvee^3 S^1$ be a one point union of three circles with the natural base point. Let a, b, c be the based oriented loops in Y tracing each of these three circles. Let X be a CW complex obtained by attaching a 2-cell e^2 by an attaching map $\varphi : S^1 \to Y$ given by $\varphi = a^{-1}bcab$. Compute the homology of X.

[**6**] Prove that the real projective space $\mathbb{R}P^n$ is orientable if and only if n is odd.

[**7**] Let X and Y be smooth vector fields on a smooth manifold M. Prove the identity

$$\mathcal{L}_X \circ \iota_Y - \mathcal{L}_Y \circ \iota_X - \iota_{[X,Y]} = [d, \iota_X \circ \iota_Y]$$

as operators acting on differential forms on M. (The bracket on the right denotes the commutator of two linear maps.)

[**8**] On a surface M with everywhere non-positive Gaussian curvature, prove that there exist no geodesic n-gons with $n < 3$. (In other words, in M, a smoothly closed geodesic or a broken geodesic with either one or two corners cannot bound a simply connected region).