GEOMETRY AND TOPOLOGY PRELIMINARY EXAM

Fall 2018

[1] Fix orientations on $S^3, \mathbb{RP}^3$ such that the covering map $p : S^3 \to \mathbb{RP}^3$ is orientation-preserving. Prove that there exists $\eta \in \Omega^3(\mathbb{RP}^3)$ such that

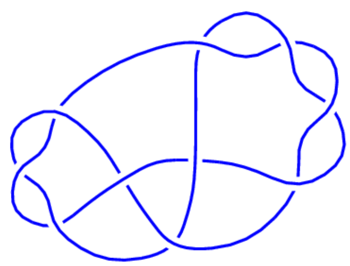
$$\int_{S^3} p^*(\eta) = 1.$$

[2] Let X be a compact surface spanning a knot given below. (Thus, the boundary of X is the given knot.) There are several possible choices. Choose one and draw its picture below by shading regions. Answer the following questions with explanation.

(1) Is X orientable?

(2) Using cell decomposition of the surface, compute the Euler characteristic $\chi(X)$ of the surface X.

(3) Let X^* be a surface without boundary obtained by gluing a disc to X along the boundary. Identify X^* (its genus and orientability)



[3] Show that if M is a compact smooth orientable surface in $\mathbb{R}^3$ that is not diffeomorphic to a sphere, then there is a point p in M at which the Gaussian curvature is negative.

[4] Let $f\colon \mathbb{C}P^n \to \mathbb{R}$ be given in terms of homogeneous coordinates by

$$f(z_0, \ldots, z_n) = \frac{\lambda_0|z_0|^2 + \ldots + \lambda_n|z_n|^2}{|z_0|^2 + \ldots + |z_n|^2},$$

where all coefficients λ_j are distinct. Show that (1) f is a smooth function and (2) find its critical points. (There are exactly $n+1$ critical points.)

[5] Prove the following Cartan formula: for any differential form ω and a vector field X on a differential manifold M^n,

$$\mathcal{L}_X\omega = i_X d\omega + d(i_X\omega).$$

[6] Define

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6$$

as a 2-form on $\mathbb{R}^6$. Show that no diffeomorphism $\phi : \mathbb{R}^6 \to \mathbb{R}^6$ which satisfies $\omega = \phi^*(\omega)$ can map the unit sphere S^5 to a sphere of radius $r \neq 1$.

[7] Consider a 3-dimensional unit cube $K = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x, y, z \le 1\}$. Construct a space M by identifying opposite faces by translation followed by rotating by 90° counter-clockwise. Introduce the cell structure on M induced from the natural cell structure of the cube.

(1) Count the number of 0-cells, 1-cells, 2-cells, and 3-cells in the space M.

(2) Calculate the homology group of M.

(3) Calculate the fundamental group of M.