Geometry and Topology Prelim, Spring 2019

1. Show that the tangent bundle TM of a differentiable manifold M is an orientable manifold, regardless of whether or not M is orientable. Moreover, TM carries a canonical orientation (as a manifold).

2. Consider the vector fields $v = \partial_x$ and $w = x\partial_y + \partial_z$ on $\mathbb{R}^3$. Let f be a function such that $L_v f = 0$ and $L_w f = 0$ everywhere on $\mathbb{R}^3$. Prove that f is a constant function.

3. Let $X = \mathbb{CP}^1$, and let $f : X \to X$ be given by

$$f([z_0, z_1]) = [\overline{z_0}, \overline{z_1}]$$

where the bar indicates complex conjugation and $[z_0, z_1]$ is the equivalence class of $(z_0, z_1) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.

   (a) Show that f is smooth.

   (b) What is the degree of f?

   (c) What are the fixed points of f?

4. Let $F\colon P \to M$ be a surjective (i.e., onto) smooth map and let α be a differential form on M. Show that $\alpha = 0$ whenever $F^*\alpha = 0$. (The converse is obvious.)
Hint: you need to use Sard's lemma.

5. Let $Z \subset \mathbb{R}^2$ be the unit circle, and consider the map

$$f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$$
$$f(x, y) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

   - Show that f is not transverse to Z.
   - Find a smooth homotopy $F : [0, 1] \times \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}^2$ such that $F(x, 0) = f$ and $F(x, 1)$ is transverse to Z. Justify your claim.

6. Let $\{U, V, W\}$ be an open cover of a space X by contractible open sets U, V, and W satisfying $U \cap V \approx S^3$, $U \cap W = \emptyset$, and $V \cap W \approx S^1$. Compute the homology of X.
Hint: first compute the reduced homology of $Y = U \cup V$.

7. The *shape operator* at a point p in a surface M is the linear map $S_p : T_pM \to T_pM$ obtained from the linearization D_pN of the Gauss map at p by identifying $T_{N(p)}S^2$ and T_pM in the 'obvious' way. (Some authors use the opposite sign convention, but that won't matter here.) A curve $\alpha : I \to M$ in a surface M is a *line of curvature* if $\alpha'(t)$ is an eigenvector of the shape operator for all $t \in I$. Show that α is simultaneously a geodesic and a line of curvature of M if and only if $\alpha(I)$ lies in a plane P that is perpendicular to M at every point in $M \cap P$.